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Jumps in layered miscible fluids

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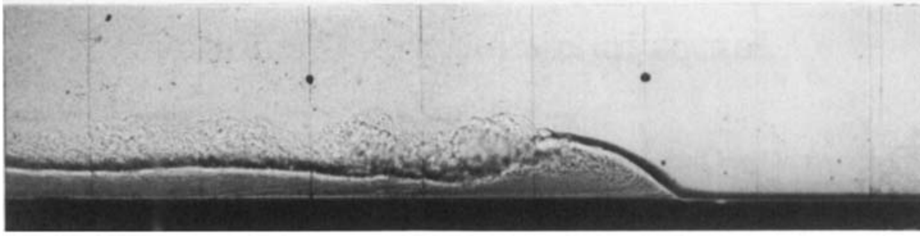
Experiments with hydraulic jumps in a layered flow with a small density difference at the interface are described. Two different configurations are examined, with shear between the fluids either upstream or downstream of the jump. It is shown that when the shear stress on all interfaces is small enough for there to be no mixing either the theory which assumes hydrostatic pressure on the face of the jump or that which assumes energy conservation in one of the layers describes the results. In the experiments for a jump in the lee of a towed obstacle this condition is always satisfied, but for a jump advancing into stationary layers it is only satisfied when the ratio of the height behind the jump to that in front is less than about 2. Beyond this limit there is mixing behind the first wave of the undular jump and the flow behaves like the head of a gravity current. The theory with energy conservation in one layer is extended to the case of a stationary jump, and for this case it is shown that for a given downstream control an approximate value of the fluid entrained can be computed.

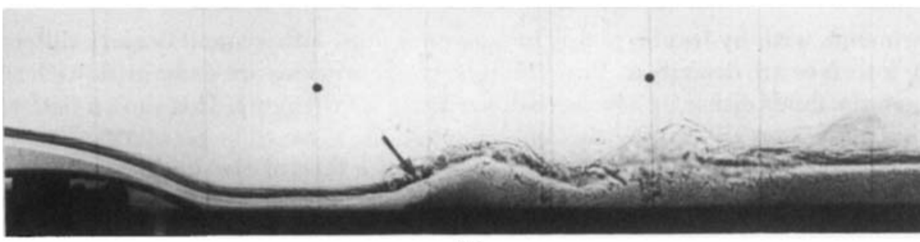
1. Introduction

Hydraulic jumps and surges (moving jumps) in open channels have considerable importance in civil engineering and have been studied for some time (Henderson 1966). In both flows air may be entrained through the air–water interface by the gravitational instabilities behind the steep face of the jump. This air is, however, detrained downstream of the abrupt change in depth. This release means that the conditions far downstream of the jump are unaffected by the air entrainment. With layered fluids such as flows in the atmosphere or in lakes or cooling ponds there are also abrupt changes in layer depths. These jumps may also be stationary or moving, but in this case fluid that is entrained is not released but remains in the flowing fluid downstream of the jump. In the atmosphere these jumps have been observed by Schweitzer (1953), Ball (1959), Lied (1966), Clarke, Smith & Reid (1981) and others.

When the depth change across the jump is small both the shear stress between the layers and the interfacial slopes are sufficiently small for the entrainment to be negligible and the jump to resemble an undular jump in an open channel. However, for large depth changes a shear-flow instability occurs. This is illustrated in figure 1. In figure 1 (*a*) a jump is advancing into a stationary fluid. The shear stress and hence the mixing is significant downstream of the maximum jump height. This implies that the conditions far downstream of the jump are affected by the mixing. In figure 1 (*b*), where an obstacle is moved through a two-layered fluid the maximum



(a)



(b)

FIGURE 1. (a) A jump advancing to the right into a layer of fluid 0.3 cm deep. The total depth y_1+y_2 of the layers is 27 cm, g' is 11.5 cm/s^2 and the gate was opened to 8 cm. In this case the velocity in both layers ahead of the jumps is zero, but the shear instability behind the head means the jump behaves as a gravity current. (b) The jump formed behind an obstacle moving to the left. In this case the mean velocity in both layers behind the jump are zero and the shear is ahead of the jump. Note that the arrow indicates the mixing of the dye in the lower layer.

shear stress and the possibility of mixing in the supercritical region occurs ahead of the jump. This affects the conditions upstream of the major abrupt change in level.

In this paper the two theories that are used for non-entraining jumps are described, and the reasons why they give similar results are discussed. Experiments are then described for the case of a jump that forms downstream of a towed obstacle and for a jump advancing into a stationary fluid. In the first case the shear stress at the interface is sufficiently small for the theory to be applicable over the range of the experimental data, while in the second the theory is only applicable until a shear-flow instability occurs. Finally, the more recent theory is applied to an entraining jump and it is shown that the condition for the jump to be stationary enables us to compute, for given downstream conditions, the amount of entrainment.

2. Theory

Long (1954) carried out an extensive experimental study of the flow of two immiscible layers over a towed obstacle. He was able to determine both analytically and experimentally the regions in which a range of flow phenomena including hydraulic jumps occurred. Yih & Guha (1955) focused on the relationship between the conditions upstream and downsteam of a jump. They considered the flow of two layers over a horizontal plane surface for the case where there is a free upper surface. Other papers (Long 1970; Hayakawa 1970; Mehrotra 1973; Mehrotra & Kelley 1973; Baines & Davies 1980) deal with the case where the upper and lower layers are confined between horizontal solid upper and lower surfaces as illustrated in figure 2. In this figure the duct depth is D and the discharge and densities in the upper and

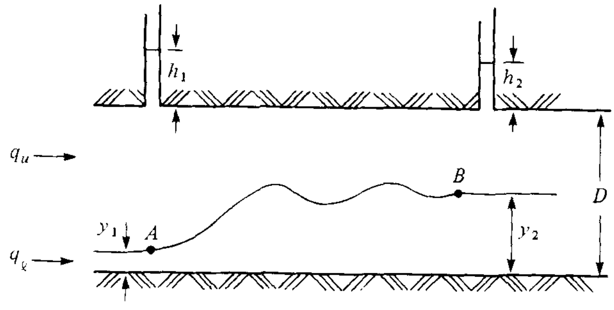


FIGURE 2. The nomenclature for the internal jump.

lower layers are q_u, q_ℓ, ρ_u and ρ_ℓ respectively, and the depths of the lower and upper layers upstream and downstream of the jump are respectively y_1, $D-y_1$, y_2 and $D-y_2$.

Yih & Guha (1955) and the other investigators referred to above assumed that the fluids did not mix, there was no shear stress at the surface between the two fluids and the mean pressure on the surface of the jump (A, B; figure 2) was $\frac{1}{2}(D-y_1+h_1+D-y_2+h_2)$, where h_1 and h_2 are the static heads on the upper surface of the duct upstream and downstream of the jump. These assumptions enable the flow-force equation to be used for each layer and a solution to be obtained.

For stationary entraining jumps in flowing layers Macagno & Macagno (1975) used Yih & Guha's relationships with a hypothesis for entrainment and subsequent mixing. They computed the energy loss for a non-mixing jump and proposed that a constant proportion α of this was the power available for entrainment. They assumed that practically all the entrainment took place over a short length at the foot of the jump and that all of the excess of production over dissipation in this region was available for entrainment. The constant α was determined from experimental measurements of hydraulic jumps in air.

The method then was to use the Yih & Guha relationships to determine the energy loss ΔE in a non-mixing jump, and then to assume that $\alpha\,\Delta E$ was the power available for entrainment. They then computed the changes that the entrainment effectively induces in the oncoming flow. This led to new inlet values of depths, discharges and densities and these were used for the final conjugate quantities.

Yih & Guha's theory has been used frequently and is well known. The more recent theory of Chu & Baddour (1977) is less well known and will be briefly described before discussing the results obtained from both theories.

A jump in two coflowing layers consists of an expansion in one layer and a similar contraction in the other. The angle of contraction is not large, and for jumps in which no mixing occurs the interface between the layers is relatively smooth. Dye-streak observations suggest that over the short distance of the jump the energy losses in the contracting layer are likely to be an order of magnitude less than that in the expanding layer. Indeed, dye placed in the contracting layer upstream of the jump remains relatively unmixed downstream of the jump, whereas dye placed in a similar position in the expanding layer becomes mixed throughout the lower layer downstream of the jump (figure 1b). Further, if wall friction is negligible, it is reasonable to assume that the energy losses in the contracting layer are small and constancy of flow force for the system of both layers. These assumptions lead to a particularly simple conservation relationship.

The case where the expanding layer is on the lower surface (a jump) is dealt with here, and the case where the expanding layer is adjacent to the upper surface (a drop) can be handled in the same simple manner.

For the jump illustrated in figure 2 the flow force per unit width is

$$\rho_u g h_1 D + \tfrac{1}{2}\rho_u g D^2 + \tfrac{1}{2}(\rho_\ell - \rho_u) g y_1^2 + \frac{\rho_\ell q_\ell^2}{y_1} + \frac{\rho_u q_u^2}{D - y_1} = S, \tag{1}$$

where S is a constant. The Bernoulli equation for the upper layer with datum taken at the bottom of the lower layer is

$$h_1 + D + \left(\frac{q_u}{D - y_1}\right)^2 \frac{1}{2g} = E_u, \tag{2}$$

where E_u is the total head of the upper layer. Eliminating h between (1) and (2), defining

$$q_r = \frac{q_u}{q_\ell}, \quad \rho_r = \frac{\rho_u}{\rho_\ell},$$

and noting that when there is no mixing these are conserved quantities, we obtain

$$\tfrac{1}{2}y_1'^2 + \frac{q_\ell^2}{g' D^3}\left[\frac{1}{y_1'} + \rho_r q_r^2 \frac{\frac{1}{2} - y_1'}{(1 - y_1')^2}\right] = S_*, \tag{3}$$

where

$$y' = \frac{y}{D}, \quad g' = \frac{(\rho_\ell - \rho_u) g}{\rho_\ell},$$

and

$$S_* = \frac{S}{(\rho_\ell - \rho_u) g D^2} - \frac{\rho_r g E_u}{g' D} + \frac{\rho_r g}{2g'}$$

is a modified flow force.

The difference ΔE in the Bernoulli constant in the lower layer on either side of the jump can then be computed using (3) and the appropriate energy equation, and the total head change Δe_T over the jump is then given by the difference in the fluxes of energy across the jump. Since there is no loss assumed in the upper layer, this is

$$\begin{aligned} |\Delta E_{T1}| &= U_\ell y_1 |\Delta E| \\ &= \frac{U_\ell^3 y_1}{2g}\left\{\frac{(y_2' - y_1')^3}{y_2'^2(y_1' + y_2')} + \frac{\rho_r q_r^2 y_1'^2}{y_1' + y_2'}\left[\frac{2 - 3y_2' + y_1'}{(1 - y_2')^2} - \frac{2 - 3y_1' + y_2'}{(1 - y_1')^2}\right]\right\}, \end{aligned} \tag{4}$$

where U_ℓ is the velocity at section 1 in the upper layer. For a very deep layer when $1 - y_1' = 1 - y_2'$,

$$|\Delta E_T| = U_\ell y_1 |\Delta E| = \frac{U_\ell^3 y_1 (y_2' - y_1')^3}{2g y_2'^2 (y_1' + y_2')} + 4\rho_r U_\ell y_1 \frac{U_u^2 (y_1' - y_2')}{2g(y_1' + y_2')}, \tag{5}$$

where U_u is the velocity in the upper layer at §1. Thus for the same y_1', y_2' and U_ℓ the upper flow reduces the loss in the lower layer. In a free-surface regime for small depth changes the small loss of energy not accounted for by friction on the lower surface is radiated away from the jump by the formation of waves. Since (5) suggests that the losses are decreased by a flow in the upper layer, it might be expected that this regime would be extended to greater height differences when the upper layer is flowing.

The conservation relationship equivalent to (3) obtained using Yih & Guha's assumption is

$$\tfrac{1}{2}y_1'^2 + \frac{q_\ell^2}{g' D^3}\left(\frac{1}{y_1'} - \frac{\rho_r q_r^2 (y_1' + y_2')}{(1 - y_1')[2 - (y_1' + y_2')]}\right) = S_y. \tag{6}$$

This equation contains variables from both sides of the jump and is therefore not a simple conservation relationship, and indeed it is this fact that leads to the problems of multiple solutions that have been explored by Yih & Guha (1955), Mehrotra (1973) and Mehrotra & Kelley (1973). Further, the head loss over the jump obtained using Yih & Guha's solution is given by Su (1976) as

$$|\Delta E_{\mathrm{T}}| = \frac{U_{\ell}^3 y_1 (y_2' - y_1')^3}{2g(y_1' + y_2')\, y_2'^2} - \rho_{\mathrm{r}} \frac{U_{\mathrm{u}}^3 (D - y_1)(y_2' - y_1')^3}{2g(2 - (y_1' + y_2'))(1 - y_2')^2}. \tag{7}$$

In this solution the upper- and lower-layer losses are independent and there is always an energy gain in the upper layer. The energy gain appears to be caused by the use of the assumption of hydrostatic pressure on the curved face of the jump, and this gain implies that the approximation slightly underestimates the force on the sloping face.

At this stage it is worth noting that for the above theory it is not necessary to assume which layer is contracting. This contrasts with the Chu & Baddour theory, where an assumption about the layers is necessary.

Before discussing the solutions, it is necessary to obtain the value of critical depths in the duct. These may be obtained by using the condition for no upstream propagation of small long waves, or, for fixed q_{r} and S_*, by determining the maximum q_ℓ from (3). This leads to

$$F_\ell^2 + \rho_{\mathrm{r}} F_{\mathrm{u}}^2 = 1, \tag{8}$$

where

$$F_\ell^2 = \frac{q_\ell^2}{g' y^3}, \quad F_{\mathrm{u}}^2 = \frac{q_{\mathrm{u}}^2}{g'(D-y)^3}.$$

Further, since in this case the jump is infinitely small (i.e. $y_1' \approx y_2' \approx y_{\mathrm{c}}$), the same conditions are obtained from both sets of assumptions.

With the Boussinesq approximation, (8) may be written as

$$q_*^2 \left[\frac{1}{y_{\mathrm{c}}'^3} + \frac{q_{\mathrm{r}}^2}{(1 - y_{\mathrm{c}}')^3} \right] = 1, \tag{9}$$

where $q_*^2 = q_\ell^2 / g' D^3$, and y_{c}' is the critical depth. This condition limits the region for which solutions of (3) or (6) are possible.

3. The jump behind the moving obstacle

3.1. *Theory*

When an obstacle on the floor at one end of a tank containing a two-layered fluid is moved then after a short time a steady jump is established (figure 1*b*). Initially there is a disturbance (Houghton & Kasahara 1968) behind the jump which will move to the wall, be reflected from it and move back to the jump where it is absorbed. After this the jump appears steady and the conditions downstream of it are no mean velocity in either layer (figure 3). Thus, if we move with the jump speed U_{w}, the discharge ratio is given by

$$q_{\mathrm{r}} = \frac{q_{\mathrm{u}}}{q_\ell} = \frac{1 - y_2'}{y_2'}. \tag{10}$$

Substituting this value into (3) yields

$$\frac{U_{\mathrm{w}}^2}{g' y_2} = \frac{y_1'(y_1' + y_2')(1 - y_1')^2}{y_2'(2y_2' - 3y_1' y_2' + y_1'^2)}. \tag{11}$$

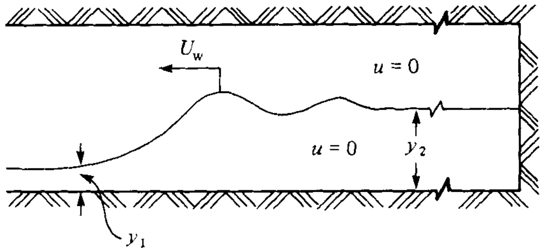


FIGURE 3. The nomenclature for the jump behind an obstacle moving to the left.

For this case the limiting critical conditions downstream of the jump are given by

$$\frac{U_w^2}{gy_{2c}} = 1 - y'_{2c}, \tag{12}$$

where y'_{2c} is the critical depth downstream of the jump. Equation (11) and the limits of its applicability, equation (12), are plotted in figure 4. Equation (10) may also be substituted into (6), and yields

$$\frac{U_w^2}{g'y'_2} = \frac{y'_2 + y'_1}{2y'_2} \Bigg/ \left[\frac{y'_2}{y'_1} + \frac{(1+y'_2)(y'_1+y'_2)}{(1-y'_2)[2-(y'_1+y'_2)]}\right]. \tag{13}$$

This is also plotted in figure 4, and it can be seen that, in spite of the difference in the form of (11) and (13), the difference in the predictions is extremely small. Discussion of this is postponed until after describing the first set of experiments.

3.2. *Experiments*

All the experiments described in this and in §4 were carried out in a flume with a length, depth and breadth of respectively 3.7, 0.5, 0.2 m, and the experiments were observed and photographed on a shadowgraph screen. The flume was filled with fresh water and the lower layer was introduced slowly under the fresh water. It flowed as a viscous gravity current along the horizontal floor. Care was taken to remove the mixed region at the head of the lower layer and to maintain a sharp interface.

Once the lower layer had reached an appropriate depth the downstream jump was produced.

For large downstream jumps this was done by towing the obstacle along the bottom of the flume through the lower layer. At the commencement of movement of the obstacle the level of the interface downstream of the obstacle dropped (Houghton & Kasahara 1968), but after a short time an equilibrium level and a downstream jump were established as in figure 3. A disturbance also propagates upstream, and this was eventually reflected off the far end of the flume. When this reflection reached the obstacle the flow became unsteady and the experiment was terminated. The movement of the jump and its depth were photographed and the depths and velocity of the jump were obtained from the negatives.

For small jumps a continuous flow was established in the lower layer of the flume, and this was suddenly stopped by either a downstream gate or a valve. The movement of the jump back along the flume was then timed and the depths measured by marking them on the shadowgraph screen.

No attempt was made to measure the velocity distribution or density distribution

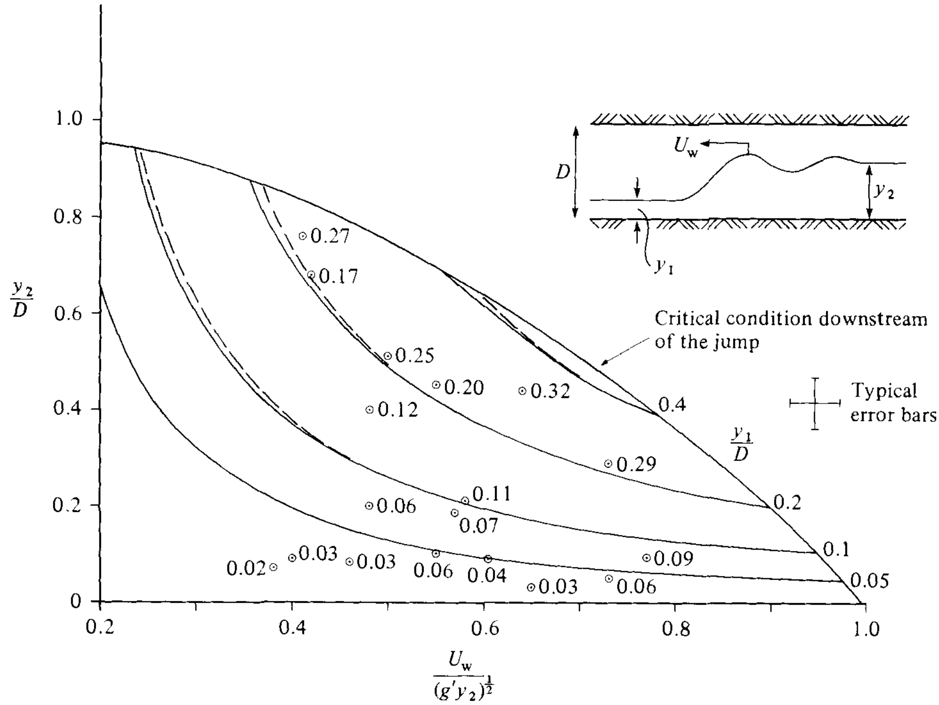


FIGURE 4. The theoretical curves and the experimental points for the jump behind a moving obstacle: ——, theory based on the energy-conserving assumption in the contracting layer; – – – –, theory based on the hydrostatic assumption (Yih & Guha 1955); ⊙ 0.03, experimental points and the experimental value of y_1/D.

in the layer prior to the jump, and it is apparent that, because of bottom friction and the shear on the upper surface, these would differ from the rectangular distributions assumed. Mixing due to Kelvin–Helmholtz instabilities in the supercritical region of the flow did not occur during the experiment.

The experimental points are plotted in figure 4, and, in view of the uncertainties discussed above and the error bands, the agreement with either theory is reasonable. This suggest that the magnitude of the change of energy (either positive or negative) in the upper layer is small. It is apparent from (7) that this is reasonable for weak jumps ($y_2'-y_1'$ small) and for the particular case of strong jumps where the velocity ahead of and relative to the jump in the upper layer is small ($u_u \ll 1$). These conditions were satisfied in this set of experiments and indeed appear to have been satisfied in the strong jump experiment carried out by Yih & Guha (1955).

4. The jump advancing into two stationary layers

4.1. *Theory*

This case is illustrated in figure 5, and if we again select axes moving with the jump velocity the discharge ratio is determined by the upstream conditions as

$$q_r = \frac{q_u}{q_\ell} = \frac{1-y_1'}{y_1'}. \tag{14}$$

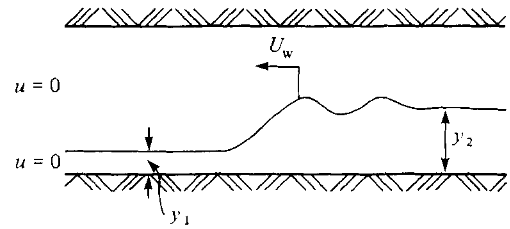


FIGURE 5. The nomenclature for the jump moving into a stationary two-layered fluid.

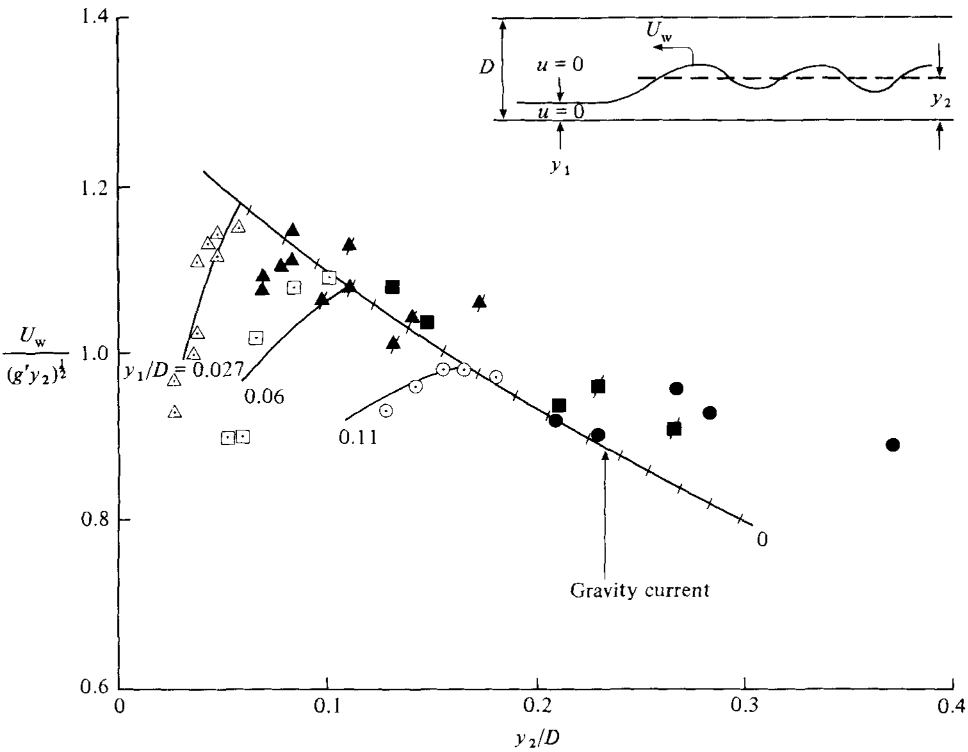


FIGURE 6. The theoretical curves and experimental points for the jump advancing into a two-layered flow. Theory: ——, theory based on energy conservation in the upper layer; -/-/-, gravity-current results (Simpson & Britter 1979). Experiment: $y_1/D = 0.027$: △, smooth waves behind the jump; ▲, broken waves behind the jump; ▲, appears as a gravity current; $y_1/D = 0.06$: ⊡, smooth waves behind the jump; ■, broken waves behind the jump; ■, appears as a gravity current; $y_1/D = 0.12$: ⊙, smooth waves behind the jump; ●, broken waves behind the jump.

Depending on the assumption adopted, this may be substituted into either (3) or (6). For the range of experiment possible either is satisfactory, and for this work (14) is substituted into (3), yielding

$$\frac{U_w^2}{gy_2} = \frac{(y_1'+y_2')(1-y_2')^2}{2y_1'-3y_1'y_2'+y_2'^2}. \tag{15}$$

This relationship is plotted in figure 6.

Critical conditions at §2 would limit the solution. However, before this limit is reached and before there is any significant difference between the results of the different assumptions, the value of y_2/y_1 becomes sufficiently large that a shear instability develops on the first wave of the undular jump. This mixing, which first

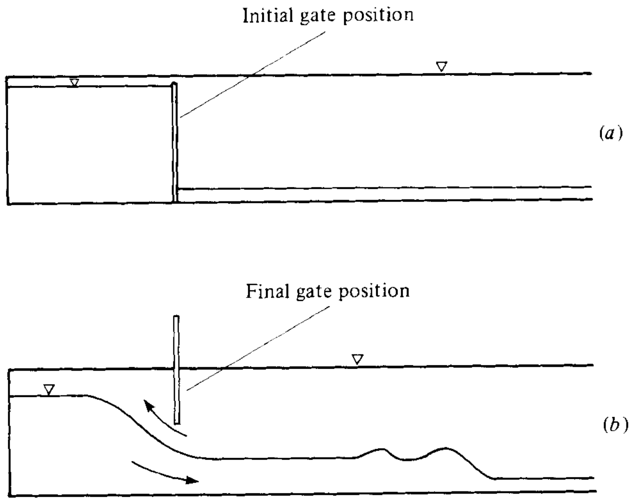


FIGURE 7. One of the methods used in obtaining a jump advancing into stationary layers: (*a*) the initial set-up; (*b*) the flow once the gate was lifted and the experiment commenced.

occurs at the crest of the wave, controls the velocity of the jump, and in the limit the flow behaves as a gravity current. It is therefore appropriate to plot the gravity-current results from Simpson & Britter (1979) on figure 6. Since the surge cannot move faster than the case where y_1 is zero, this curve provides the upper limit to $U_w/(g'y_2)^{\frac{1}{2}}$.

4.2. *Experiments*

The same technique was used as in §3.2, but in this case the upstream jump was observed. For depths such that $y_1 \approx y_2$ the velocity of the surge was obtained by timing the start of the movement of dye patches in the lower layer.

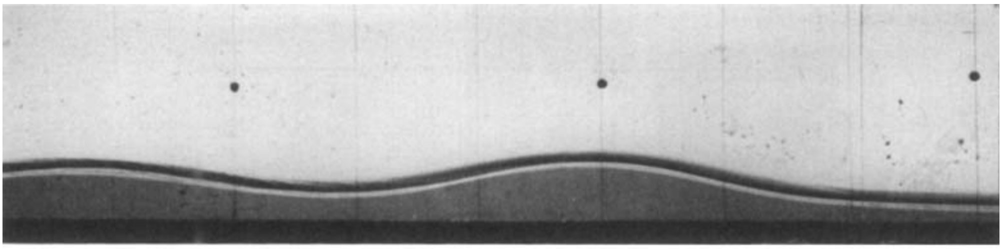
For large depth differences two techniques were used. For small total depths D the jump that moved ahead of the towed obstacle was observed. However, for large total depths this surge moved ahead of the obstacle so rapidly that steady-state measurements could not be obtained.

For these large depths the flume was set up as in figure 7(*a*), and when the gate was partially removed (figure 7*b*) a steady exchange flow controlled by the final gate position commenced. This enabled satisfactory steady-state observations to be made.

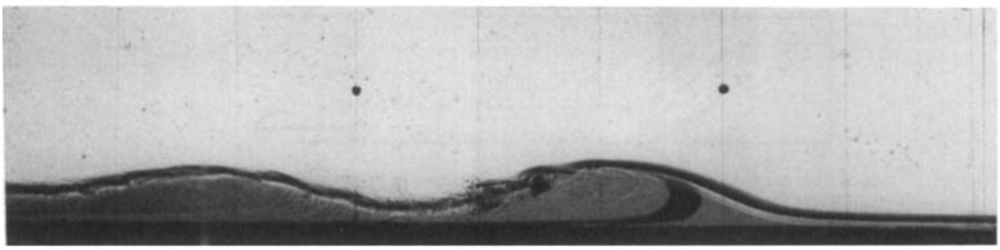
In all the experiments the density difference between the two layers compared with the density difference between the upper fluid and air was small enough for the free surface in the experiment to be treated as the solid surface used in the theory.

The change in the character of the flow is best illustrated in figure 8. For the flows in this figure the depth of flow was kept at 27 cm, the density difference was maintained giving $g' = 11.5\ \mathrm{cm/s^2}$, and the gate was raised to 8 cm above the floor. For a depth ahead of the jump of 1.75 cm the jump was smooth and undular (figure 8*a*). However, when the depth was reduced to 0.75 cm the shear instability on the rear face of the first wave became pronounced and there was considerable mixing. This is shown by the behaviour of the dye that was originally in the stationary lower layer of figure 8(*b*). Finally, when the lower layer was reduced to 0.3 cm the flow appears as a well-defined gravity current (figure 1*a*).

Experiments were carried out with the values of $y_1' = 0.027$, 0.06 and 0.12. For the very small depth changes in which dye was used to determine the surge velocity the experimental points lie below the theoretical curve. It is believed that this is because



(a)



(b)

FIGURE 8. The jumps produced by the method of figure 7. In both cases the total depth y_1+y_2 of the layers is 27 cm, g' is 11.5 cm/s^2 and the gate was opened to 8 cm: (a) $y_1 = 1.75$ cm; (b) $y_1 = 0.75$ cm.

y_1'	Maximum value of y_2/y_1 with no mixing	Minimum value of y_2/y_1 with mixing
0.027	2.2	2.7
0.06	1.7	2.2
0.011	1.6	1.9

TABLE 1

of viscous effects. When the undular jump was smooth and sufficiently large for the mean downstream depth to be measured the agreement between theory and experiment is reasonable.

Table 1 gives for each y_1' the experimentally determined maximum value of y_2/y_1 for no observable mixing behind the first wave and the minimum value of y_2/y_1 for observable mixing.

The lower minimum at the larger values of y_1' is presumably due to the greater shear in these cases. It is to be noted that after the mixing first occurs (which is a value of y_2/y_1 of approximately two) the value of $U_w/(gy_2)^{\frac{1}{2}}$ is dominated by this mixing, and in many cases the flow resembles a gravity current.

5. The stationary entraining internal jump

5.1. *Theory*

As the height of the towed obstacle in the first set of experiments increases, Kelvin–Helmholtz instabilities occur and mixing will take place prior to the jump conditions (as determined by either set of assumptions) being satisfied. Indeed the

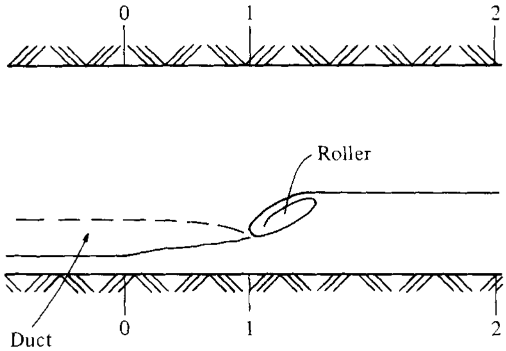


FIGURE 9. The nomenclature for the stationary jump: ----, the streamline dividing the entrained fluid and the freestream.

amount of mixing taking place is that quantity necessary to change the upstream flow values to those which satisfy the jump conditions. This implies that the amount of mixing is determined by these conditions.

Where flow exits from a duct (figure 9) the shear layer immediately downstream of the duct causes this mixing, and for a given downstream control the jump is stationary. For this case when there is no flow in the upper layer the control of the mixing was investigated by Wilkinson & Wood (1971) and Stefan & Hayakawa (1972). The use of the energy-conservation assumption enables this work to be extended to the case where there is a flow in the upper layer. For this flow (figure 9), provided that an allowance is made for the change in density and the discharge in the lower layer, (1) is still applicable. Further, the Bernoulli equation is applicable between sections 0 and 2 on the streamlines which do not enter the jump. Thus, with these modifications, (3) is still applicable.

If the density differences are sufficiently small for the Boussinesq approximation to be reasonable, then the equation of continuity of the modified gravity g' may be written as

$$q_\ell g' = q_\Delta, \tag{16}$$

where q_Δ, the flux of g', is a constant. Substituting into (3) and defining the discharges in the layers at sections 0 and 2 as $q_{\ell 0}$, q_{u0}, and $q_{\ell 2}$, q_{u2} respectively, (3) becomes

$$\frac{\frac{1}{2}q_\Delta y_2^2}{q_{\ell 2}} + \frac{q_{\ell 1}^2}{y_2} + \frac{q_{u2}^2(\frac{1}{2}D - y_2)}{(D-y_2)^2} = \frac{\frac{1}{2}q_\Delta y_0^2}{q_{\ell 0}} + \frac{q_{\ell 0}^2}{y_0} + \frac{q_{u0}^2[\frac{1}{2}D - y_0]}{(D-y_0)^2}. \tag{17}$$

Downstream of §2 the flow may be controlled by a weir or contraction, and this will determine the particular conditions at §2. It will be assumed that the condition at this section is given by

$$F_{\ell 2}^2 + F_{u2}^2 = \frac{q_{\ell 2}^3}{q_\Delta y_2^3} + \frac{q_u^2 q_{\ell 2}}{q_\Delta (D-y_2)^3} = A. \tag{18}$$

Finally, in a duct we have

$$q_{\ell 0} + q_{u0} = q_{\ell 2} + q_{u2}. \tag{19}$$

The equations can be solved, but the results for the most interesting case where the duct is very deep can be obtained in a very simple manner.

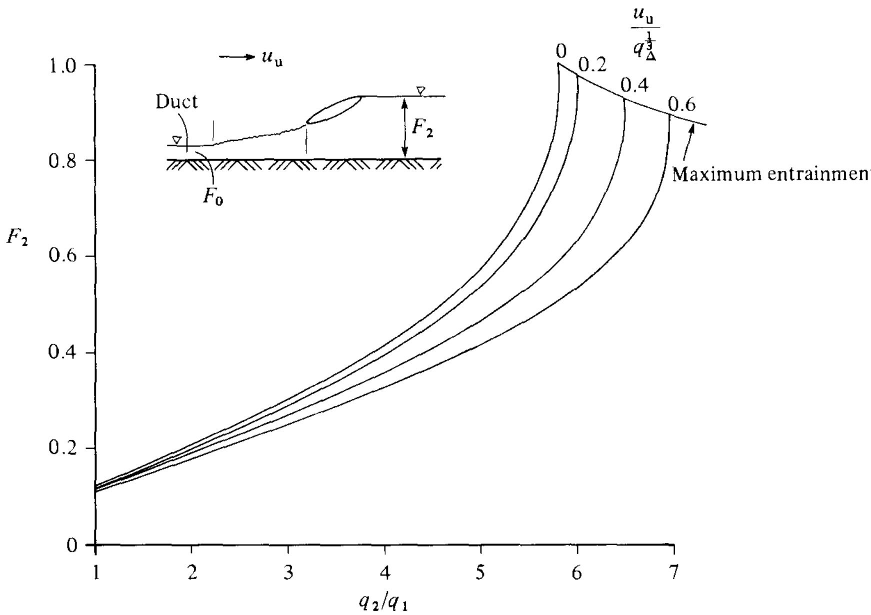


FIGURE 10. The results for the stationary jump with $D = \infty$. The ratio of the flow in the lower layer downstream of the jump to that upstream of the jump for a range of downstream Froude number. The upstream Froude number is 25 and the non-dimensional velocity $U_u/q^{\frac{1}{3}}$ in the upper layer is varied from 0 to 0.6.

Equation (18) can be written as

$$\frac{q_{\ell 2}^3}{q_\Delta y_2^3}\left[1+\frac{U_{u2}^2 y_2}{U_2^2(D-y_2)}\right] = A \tag{20}$$

and

$$\frac{U_{u2}^2 y_2}{U_{\ell 2}^2(D-y_2)} \ll 1,$$

then

$$\frac{q_\ell^3}{q_\Delta y_2^3} = A = F_2^2. \tag{21}$$

For a very deep duct where $U_{u2} \approx U_{u0}$, (17) becomes

$$q_{\ell 2}\left[\frac{1+2F_2^2}{2F_2^{\frac{4}{3}}}-\frac{U_{u0}^2}{q_\Delta^{\frac{2}{3}}F_2^{\frac{2}{3}}}\right] = q_{\ell 0}\left[\frac{1+2F_0^2}{2F_0^{\frac{4}{3}}}-\frac{U_{u0}^2}{q_\Delta^{\frac{2}{3}}F_0^{\frac{2}{3}}}\right]. \tag{22}$$

For given upstream conditions F_0, q_Δ and a given flow U_{u0} in the upper layer, the Froude number F_2 in the subcritical region determines the discharge ratio $q_{\ell 2}/q_{\ell 0}$ in the lower layer uniquely. The maximum entrainment can be obtained by differentiating (22), and this maximum occurs when

$$F_2^2 = \left[1+\frac{U_{u0}^2}{q_\Delta^{\frac{2}{3}}}\right]^{-1}. \tag{23}$$

Equation (22) is plotted for an upstream Froude number of 25 and for a range of values of $U_{u0}/q_\Delta^{\frac{1}{3}}$ in figure 10, and the form of the curve is interesting. For the case where $U_{u0}/q_{\Delta}^{\frac{1}{3}}$ is zero, Wilkinson & Wood (1971) by using a weir as a control were able to verify the shape of the curve. Further dye streaks in the upper layer showed that the decrease in entrainment with decreasing F_2 was caused by the roller region in figure 2 extending towards the inlet and covering a greater length of the entrainment zone.

It is apparent that for a fixed downstream Froude number F_2 a velocity in the upper layer forces the roller region downstream and by exposing more of the entraining zone increases the total entrainment.

The trend agrees with the few observations available. However, it was observed by Bewick (1974) that the mixing downstream of the roller region was incomplete and that the velocity and density distribution in the region varies with both the downstream and upstream Froude numbers and the value of $U_{u0}/q_{\Delta}^{\frac{1}{3}}$. The shapes of these distributions would need to be measured and used for complete verification of this theory.

6. Discussion

In the experimental range the two theories give similar results. This suggests that within this range the effects of the shear and of the dynamic portion of the pressure on the interface both of which are neglected in the theory of Yih & Guha (1955) are indeed small. This implies that any energy change in the upper layer is small, and thus it is not surprising that the Chu & Baddour (1977) theory, which assumes no energy change, gives similar results.

Both theories break down when the shear on the interface becomes large. The condition for small shear on the interface is different for each of the boundary conditions discussed. For the hump in the lee of a towed obstacle, this condition was always satisfied. However, for the jump advancing into stationary layers it was only satisfied when y_2/y_1 was less than a particular value, and for the stationary jump the condition was satisfied when the downstream control conditions exactly satisfied the no-mixing jump equation. When there is entrainment then for this latter case an approximate estimate of the amount of entrainment can be obtained by computing the entrainment necessary to change the upstream conditions to those which satisfy the jump relationship. This is carried out in a very simple manner if the assumption of energy conservation in the contracting layer is used.

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